

THE SZEGŐ CLASS WITH A POLYNOMIAL WEIGHT

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ABSTRACT. Let p be a trigonometric polynomial, nonnegative on the unit circle \mathbb{T} . We say that a measure σ on \mathbb{T} belongs to the polynomial Szegő class, if $d\sigma(e^{i\theta}) = \sigma'_{ac}(e^{i\theta}) d\theta + d\sigma_s(e^{i\theta})$, σ_s is singular, and

$$\int_0^{2\pi} p(e^{i\theta}) \log \sigma'_{ac}(e^{i\theta}) d\theta > -\infty$$

For the associated orthogonal polynomials $\{\varphi_n\}$, we obtain pointwise asymptotics inside the unit disc \mathbb{D} . Then we show that this asymptotics holds in L^2 -sense on the unit circle. As a corollary, we get existence of certain modified wave operators.

INTRODUCTION

Let σ be a non-trivial Borel probability measure on the unit circle $\mathbb{T} = \{z : |z| = 1\}$. Consider orthonormal polynomials $\{\varphi_n\}$ with respect to the measure,

$$\int_{\mathbb{T}} \varphi_n \overline{\varphi_m} d\sigma = \delta_{nm}$$

where δ_{nm} is the Kronecker's symbol. Sometimes, it is more convenient to work with monic orthogonal polynomials $\{\Phi_n\}$, $\Phi_n(z) = z^n + a_{n,n-1}z^{n-1} + \dots + a_{n,0}$. These polynomials satisfy

$$\int_{\mathbb{T}} \Phi_n \overline{\Phi_m} d\sigma = c_n \delta_{nm}$$

with $c_n = \|\Phi_n\|_{\sigma}^2 = \int_{\mathbb{T}} |\Phi_n|^2 d\sigma$.

It is well-known [4, 7] that polynomials $\{\Phi_n\}$ generate a sequence $\{\alpha_n\}$, $|\alpha_n| < 1$, of the so-called Verblunsky coefficients through recurrence relations

$$\begin{aligned} \Phi_{n+1}(z) &= z\Phi_n(z) - \bar{\alpha}_n \Phi_n^*(z), \Phi_0(z) = 1 \\ \Phi_{n+1}^*(z) &= \Phi_n^*(z) - \alpha_n z \Phi_n(z), \Phi_0^*(z) = 1 \end{aligned}$$

where $\Phi_n^*(z) = z^n \overline{\Phi_n(1/\bar{z})}$. Conversely, the measure σ (and polynomials $\{\varphi_n\}$) are completely determined by the sequence $\{\alpha_k\}$ of its Verblunsky coefficients. Hence, it is natural to study the sequence $\{\alpha_k\}$ and polynomials $\{\varphi_n\}$ in terms of σ and vice versa.

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We say that σ is a Szegő measure ($\sigma \in (\mathcal{S})$, for brevity), if $d\sigma = d\sigma_{ac} + d\sigma_s = \sigma'_{ac} dm + d\sigma_s$ and the density σ'_{ac} of the absolutely continuous part is such that

$$\int_{\mathbb{T}} \log \sigma'_{ac} dm > -\infty$$

Here, the singular part of σ is denoted by σ_s , and m is the probability Lebesgue measure on \mathbb{T} , $dm(t) = dt/(2\pi it) = 1/(2\pi) d\theta$, $t = e^{i\theta} \in \mathbb{T}$.

The following results are classical.

Theorem 0.1 ([4, 8]). *The following assertions are equivalent*

- i) *the sequence α is in $l^2(\mathbb{Z}_+)$,*
- ii) *the measure σ belongs to the Szegő class,*
- iii) *analytic polynomials are not dense in $L^2(\sigma)$.*

The last statement of the theorem can be made more precise. Namely, we have

$$(0.1) \quad d(\mathcal{P}_1, 0)_{L^2(\sigma)}^2 = \inf_{f \in \mathcal{P}_1} \|f\|_{\sigma}^2 = \exp \int_{\mathbb{T}} \log \sigma'_{ac} dm$$

where \mathcal{P}_1 is the set of analytic polynomials f with the property $f(0) = 1$.

If $\sigma \in (\mathcal{S})$, we define a function D in the Hardy class $H^2(\mathbb{D})$ on the unit disk $\mathbb{D} = \{z : |z| < 1\}$ as

$$(0.2) \quad D(z) = \exp \left(\frac{1}{2} \int_{\mathbb{T}} \frac{t+z}{t-z} \log \sigma'_{ac}(t) dm(t) \right)$$

Theorem 0.2 ([4, 8]). *Let $\sigma \in (\mathcal{S})$. Then*

$$\lim_{n \rightarrow \infty} D(z) \varphi_n^*(z) = 1$$

for every $z \in \mathbb{D}$, and, moreover,

$$\lim_{n \rightarrow \infty} \int_{\mathbb{T}} |D \varphi_n^* - 1|^2 dm = 0$$

that is, the convergence is in the L^2 -sense on the unit circle.

A modern presentation and recent advances in this direction can be found in [5, 7].

It seems interesting to obtain similar results for different classes of measures. Consider a trigonometric polynomial p with the property $p(t) \geq 0, t \in \mathbb{T}$. Without loss of generality we can assume it is in the form

$$(0.3) \quad p(t) = \prod_{k=1}^N |t - \zeta_k|^{2\kappa_k}$$

where $\{\zeta_k\}$ are points on \mathbb{T} and κ_k are their “multiplicities”. We say that σ is in the polynomial Szegő class (i.e., σ is a (pS)-measure or $\sigma \in (\text{pS})$, to be brief), if

$d\sigma = \sigma'_{ac} dm + d\sigma_s$, σ_s being the singular part of the measure, and

$$(0.4) \quad \int_{\mathbb{T}} p(t) \log \sigma'_{ac}(t) dm(t) > -\infty$$

We give counterparts of Theorems 0.1 and 0.2 for orthogonal polynomials with respect to (pS)-measures in the next section. Then we construct modified wave operators for the corresponding CMV-representations and obtain relations similar to (0.1).

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1. RESULTS

1.1. We fix the polynomial p (see (0.3)) for the rest of this paper. For the sake of transparency, assume $\kappa_k = 1$; the discussion of the general case follows the same lines.

Let \mathcal{C} and \mathcal{C}_0 be the CMV-representations connected to σ and m (see [1, 7, Ch. 4]), and $\text{rank}(\mathcal{C} - \mathcal{C}_0) < \infty$. Recall that the function D appearing in (0.2) can be represented as

$$D(z) = \exp \left(t_0 + \sum_{k=1}^{\infty} \frac{\text{tr}(\bar{\mathcal{C}}^k - \bar{\mathcal{C}}_0^k)}{k} z^k \right)$$

Here,

$$t_0 = \sum_k \log \rho_k = \sum_k \log(1 - |\alpha_k|^2)^{1/2}$$

and $\{\alpha_k\}$ are the Verblunsky coefficients corresponding to σ .

Furthermore, the polynomial p (0.3) defines an analytic polynomial P via the formulas

$$p_1 = 2P_+p, \quad P'(t) = \frac{p_1(t) - p_1(0)}{t}$$

where $P_+ : L^2(\mathbb{T}) \rightarrow H^2(\mathbb{D})$ is the Riesz projector [3, Ch. 3].

Lemma 1.1. *Let $\text{rank}(\mathcal{C} - \mathcal{C}_0) < \infty$. Then*

$$(1.1) \quad \int_{\mathbb{T}} p \log \sigma'_{ac} dm = a_0 t_0 + \text{Re tr}(P(\mathcal{C}) - P(\mathcal{C}_0))$$

where

$$a_0 = p_1(0) = 2 \int_{\mathbb{T}} p dm$$

We denote the right-hand side of equality (1.1) by $\Psi(\mathcal{C})$ and we rewrite it in a slightly different form. To this end, we consider the shift $S : l^2(\mathbb{Z}_+) \rightarrow l^2(\mathbb{Z}_+)$, given by $Se_k = e_{k+1}$. For a bounded operator A on $l^2(\mathbb{Z}_+)$, we look at $\tau(A) =$

S^*AS . In particular, the matrix of $\tau^k(A)$, $k \in \mathbb{Z}_+$, is obtained from the matrix of A by dropping the first k rows and columns. Going back to (1.1), we get that

$$\begin{aligned}\Psi(\mathcal{C}) &= \sum_{k=0}^{\infty} \{a_0 \log \rho_k + \operatorname{Re}((P(\mathcal{C}) - P(\mathcal{C}_0))e_k, e_k)\} \\ &= \sum_{k=0}^{2N+1} \{a_0 \log \rho_k + \operatorname{Re}((P(\mathcal{C}) - P(\mathcal{C}_0))e_k, e_k)\} + \sum_{k=2N+2}^{\infty} \psi \circ \tau^k(\mathcal{C})\end{aligned}$$

where $\psi(\mathcal{C}) = a_0 \log \rho_{2N+2} + \operatorname{Re}((P(\mathcal{C}) - P(\mathcal{C}_0))e_{2N+2}, e_{2N+2})$.

Repeating word-by-word arguments from [6], Lemma 3.1, we see that there exists a function γ , depending on $l = 4N + 5$ arguments, such that

$$\psi(x_1, \dots, x_l) = \eta(x_1, \dots, x_l) - \gamma(x_2, \dots, x_l) + \gamma(x_1, \dots, x_{l-1})$$

and $\eta(x_1, \dots, x_l) \leq 0$ for any collection (x_1, \dots, x_l) . Now, we put

$$\begin{aligned}\Phi(\mathcal{C}) &= \int_{\mathbb{T}} p(t) \log \sigma'_{ac}(t) dm(t) \\ \tilde{\Psi}(\mathcal{C}) &= \sum_{k=0}^{2N+1} \{a_0 \log \rho_k + \operatorname{Re}((P(\mathcal{C}) - P(\mathcal{C}_0))e_k, e_k)\} \\ &\quad + \sum_{k=2N+2}^{\infty} \eta \circ \tau^k(\mathcal{C}) + \gamma \circ \tau^{2N+2}(\mathcal{C})\end{aligned}$$

Consequently, for a CMV-representation \mathcal{C} having $\operatorname{rank}(\mathcal{C} - \mathcal{C}_0) < \infty$, equality (1.1) reads as

$$\Phi(\mathcal{C}) = \Psi(\mathcal{C}) = \tilde{\Psi}(\mathcal{C})$$

The following theorem holds.

Theorem 1.2 ([6, Theorem 1.4]). *A measure σ is polynomially Szegő (see (0.4)) if and only if $\tilde{\Psi}(\mathcal{C}) > -\infty$. Moreover, in this case $\Phi(\mathcal{C}) = \tilde{\Psi}(\mathcal{C})$.*

1.2. We turn now to the description of asymptotical properties of orthogonal polynomials for (pS)-measures. Consider a modified Schwarz kernel

$$K(t, z) = \frac{t + z}{t - z} \frac{q(t)}{q(z)}$$

where $q(t) = C(\prod_k (t - \zeta_k)^2)/t^N$, and the constant C , $|C| = 1$, is chosen in a way that $q(t) \in \mathbb{R}$ for $t \in \mathbb{T}$ (i.e., $C = (\prod_k (-\zeta_k))^{-1}$). Furthermore, define

$$\begin{aligned}\tilde{D}(z) &= \exp \left(\frac{1}{2} \int_{\mathbb{T}} K(t, z) \log \sigma'_{ac}(t) dm(t) \right) \\ \tilde{\varphi}_n^*(z) &= \exp \left(\int_{\mathbb{T}} K(t, z) \log |\varphi_n^*(t)| dm(t) \right)\end{aligned}$$

The functions $\{\tilde{\varphi}_n^*\}$ are called (reversed) modified orthogonal polynomials with respect to σ . It can be readily seen that $|\tilde{D}|^2 = \sigma'_{ac}$ and $|\tilde{\varphi}_n^*| = |\varphi_n^*| = |\varphi_n|$ a.e. on \mathbb{T} . Furthermore, we see that $\tilde{\varphi}_n^* = \psi_n \varphi_n^*$, where

$$(1.2) \quad \begin{aligned} \psi_n(z) &= \exp \left(\int_{\mathbb{T}} \frac{t+z}{t-z} \left(\frac{q(t)}{q(z)} - 1 \right) \log |\varphi_n^*(t)| dm(t) \right) \\ &= \exp \left(A_{0n} + \sum_{k=1}^N \left(A_{kn} \frac{z + \zeta_k}{z - \zeta_k} + B_{kn} \left\{ \frac{z + \zeta_k}{z - \zeta_k} \right\}^2 \right) \right) \end{aligned}$$

and $A_{0n}, B_{kn} \in i\mathbb{R}$, $A_{kn} \in \mathbb{R}$. The coefficients $\{A_{0n}, A_{kn}, B_{kn}\}_{k,n}$ can be expressed in a closed form through $\{\alpha_k\}$.

Theorem 1.3. *Let $\sigma \in (\text{pS})$. Then*

i) *for any $z \in \mathbb{D}$,*

$$\lim_{n \rightarrow \infty} \tilde{D}(z) \tilde{\varphi}_n^*(z) = \lim_{n \rightarrow \infty} \tilde{D}(z) \psi_n(z) \varphi_n^*(z) = 1$$

ii) *we also have*

$$\lim_{n \rightarrow \infty} \int_{\mathbb{T}} |\tilde{D} \tilde{\varphi}_n^* - 1|^2 dm = 0$$

The proof of the first claim of the theorem is partially based on the sum rules proved in Theorem 1.2. One of the main facts leading to the second claim is that

$$\lim_{n \rightarrow \infty} \int_I |\tilde{D} \tilde{\varphi}_n^* - 1|^2 dm = 0$$

for any closed arc $I \subset \mathbb{T}$ that does not contain points $\{\zeta_k\}$. We prove the latter relation observing that, for small $\varepsilon > 0$,

$$|\tilde{D} \tilde{\varphi}_n^*(z)| \leq \frac{C(\varepsilon)}{\sqrt{1 - |z|}}$$

for $z \in \mathbb{D} \setminus (\cup_k B_\varepsilon(\zeta_k))$, $B_\varepsilon(\zeta) = \{z : |z - \zeta| < \varepsilon\}$.

1.3. We now use asymptotics described in the last subsection, to construct modified wave operators.

Let $\mathcal{F}_0 : L^2(m) \rightarrow l^2(\mathbb{Z}_+)$, $\mathcal{F} : L^2(\sigma) \rightarrow l^2(\mathbb{Z}_+)$ be the Fourier transforms associated to the CMV-representations \mathcal{C} and \mathcal{C}_0 , see [7, Ch. 4]. Recall that

$$\mathcal{C} = \mathcal{F} z \mathcal{F}^{-1}, \quad \mathcal{C}_0 = \mathcal{F}_0 z \mathcal{F}_0^{-1}$$

Theorem 1.4. *Let $\sigma \in (\text{pS})$. The limits*

$$(1.3) \quad \tilde{\Omega}_\pm = \text{s-lim}_{n \rightarrow \pm\infty} e^{W(2n, \mathcal{C})} \mathcal{C}^n \mathcal{C}_0^{-n}$$

exist. Here

$$W(C, n) = A_{0n} + \sum_{k=1}^N \left(A_{kn} \frac{\mathcal{C} + \zeta_k}{\mathcal{C} - \zeta_k} + B_{kn} \left\{ \frac{\mathcal{C} + \zeta_k}{\mathcal{C} - \zeta_k} \right\}^2 \right)$$

and coefficients $\{A_{0n}, A_{kn}, B_{kn}\}$ are defined in (1.2). We also have

$$\mathcal{F}^{-1}\tilde{\Omega}_+\mathcal{F}_0 = \chi_{E_{ac}} \frac{1}{\tilde{D}}, \quad \mathcal{F}^{-1}\tilde{\Omega}_-\mathcal{F}_0 = \chi_{E_{ac}} \frac{1}{\tilde{\bar{D}}}$$

where $E_{ac} = \mathbb{T} \setminus \text{supp } \sigma_s$.

As a simple corollary, we get the existence of modified wave operators for the pair $(\mathcal{C}_0, \mathcal{C})$. The natural (and open) question is to prove modified wave operators for the pair $(\mathcal{C}, \mathcal{C}_0)$.

1.4. We now briefly discuss a variational problem related to (pS)-measures. We put \mathcal{P}'_0 to be the set of polynomials g analytic on \mathbb{D} with the property $g \neq 0$ on \mathbb{D} and $g(0) > 0$. Furthermore, for a $g \in \mathcal{P}'_0$, we set

$$\lambda(g) = \exp \left(\int_{\mathbb{T}} p \log |g| \, dm \right)$$

and define $\mathcal{P}'_1 = \{g : g \in \mathcal{P}'_0, \lambda(g) = 1\}$.

Theorem 1.5. *Let $d\sigma = w \, dm + d\sigma_s$. Then*

$$\begin{aligned} (1.4) \quad \exp \left(\int_{\mathbb{T}} p \log \frac{w}{p} \, dm \right) &\leq \inf_{g \in \mathcal{P}'_1} \|g\|_{\sigma}^2 = \inf_{\substack{g \in \mathcal{P}'_0, \\ \|g\|_{\sigma} \leq 1}} \frac{1}{|\lambda(g)|^2} \\ &\leq \exp \left(\int_{\mathbb{T}} p \log w \, dm \right) \end{aligned}$$

Remind that σ is a Szegő measure if and only if the system $\{e^{iks}\}_{k \in \mathbb{Z}}$ is uniformly minimal in $L^2(\sigma)$. Saying that σ is a (pS)-measure translates into the uniform minimality of another system, $\{e^{ik\nu(s)}\}_{k \in \mathbb{Z}}$, in the same space $L^2(\sigma)$. Above,

$$\nu(s) = C_0 \int_0^s p(e^{is'}) \, ds'$$

where $s, s' \in [0, 2\pi]$ and the constant C_0 comes from the condition $C_0 \int_{\mathbb{T}} p \, dm = 1$, see [6], Lemma 2.2. It seems to be an interesting question to translate these results to the language of Gaussian processes [2].

It was proved recently in [7, Ch. 2] that $\sigma \in (\text{p}_0\text{S})$ with

$$p_0(t) = \frac{1}{2} |1 - t|^2 = 1 - \cos \theta$$

if and only if $\{\alpha_k\} \in l^4$ and $\{\alpha_{k+1} - \alpha_k\} \in l^2$ (above, $t = e^{i\theta}$). Theorems 1.2–1.5 readily apply to this special case. In particular, we have

$$K_0(t, z) = \frac{t+z}{t-z} \frac{(t-1)^2}{t} \frac{z}{(1-z)^2}$$

$$\tilde{D}_0(z) = \exp \left(\frac{1}{2} \int_{\mathbb{T}} K_0(t, z) \log \sigma'_{ac}(t) dm(t) \right)$$

and

$$\psi_n(z) = \exp \left(A_n \left\{ \frac{1+z}{1-z} - 1 \right\} + B_n \left\{ \left(\frac{1+z}{1-z} \right)^2 - 1 \right\} \right)$$

where

$$A_n = \sum_{k=0}^n \log(1 - |\alpha_k|^2)^{1/2}, \quad B_n = \frac{i}{4} \operatorname{Im} \left(\alpha_0 - \sum_{k=1}^n \bar{\alpha}_{k-1} \alpha_k \right)$$

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